ORKED XAMPLE 17 Find all values of n such that $(-\sqrt{3}+i)^n + (-\sqrt{3}-i)^n = 0$.

THINK	WRITE
1 Express the complex number $-\sqrt{3}+i$ in polar form (see Worked example 11b).	$-\sqrt{3}+i = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$
2 The complex number $-\sqrt{3}-i$ is the conjugate. Express $-\sqrt{3}-i$ in polar form.	$-\sqrt{3}-i = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$
3 Express the equation in polar form.	$(-\sqrt{3}+i)^n + (-\sqrt{3}-i)^n = 0$ $\left(2\operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^n + \left(2\operatorname{cis}\left(-\frac{5\pi}{6}\right)\right)^n = 0$
4 Use de Moivre's theorem.	$2^n\operatorname{cis}\left(\frac{5\pi n}{6}\right) + 2^n\operatorname{cis}\left(-\frac{5\pi n}{6}\right) = 0$
5 Take out the common factor and expand $\operatorname{cis}(\theta)$.	$2^n\left(\operatorname{cis}\left(\frac{5\pi n}{6}\right) + \operatorname{cis}\left(-\frac{5\pi n}{6}\right)\right) = 0$ $\cos\left(\frac{5\pi n}{6}\right) + i\sin\left(\frac{5\pi n}{6}\right) + \cos\left(-\frac{5\pi n}{6}\right) + i\sin\left(-\frac{5\pi n}{6}\right) = 0$
6 Use the trigonometric results for functions of negative angles and simplify.	Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, $2\cos\left(\frac{5\pi n}{6}\right) = 0$
7 Use the formula for the general solutions of trigonometric equations.	$\cos\left(\frac{5\pi n}{6}\right) = 0$ $\frac{5\pi n}{6} = \frac{(2k+1)\pi}{2}$ where $k \in Z$
8 Solve for n and state the final answer.	$n = \frac{3(2k+1)}{5}$ where $k \in Z$.

ERCISE 3.3 Complex numbers in polar form

PRACTISE

1 WE11 Convert each of the following complex numbers to polar form.

a $1+\sqrt{3}i$ b $-1+i$ c $-2-2\sqrt{3}i$
d $\sqrt{3}-i$ e 4 f $-2i$

2 Convert each of the following complex numbers to polar form.

a $\sqrt{3}+i$ b $-1+\sqrt{3}i$ c $-\sqrt{3}-i$
d $2-2i$ e -7 f $5i$

3 WE12 a Convert $4\operatorname{cis}\left(-\frac{\pi}{3}\right)$ into rectangular form.

b Convert $8\operatorname{cis}\left(-\frac{\pi}{2}\right)$ into rectangular form.

4 a Convert $6\sqrt{2}\,\text{cis}(-135°)$ into $x + yi$ form.

b Convert $5\,\text{cis}(126°52')$ into $a + bi$ form.

5 If $u = 6\,\text{cis}\left(-\frac{\pi}{3}\right)$, find $\overline{u}^{-1}$, giving your answer in rectangular form.

6 If $u = \frac{\sqrt{2}}{4}\,\text{cis}\left(\frac{3\pi}{4}\right)$, find $\frac{1}{u}$, giving your answer in rectangular form.

7 If $u = 4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$, find each of the following, giving your answers in rectangular form.

a uv

b $\frac{u}{v}$

8 If $u = 4\,\text{cis}\left(\frac{\pi}{3}\right)$ and $v = \frac{1}{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)$ find each of the following, giving your answers in rectangular form.

a uv

b $\frac{u}{v}$

9 If $u = -1 - i$, find:

a $\text{Arg}(u^{10})$

b u^{10}, giving your answer in rectangular form.

10 Simplify $\frac{(-1+i)^6}{(\sqrt{3}-i)^4}$, giving your answer in rectangular form.

11 a Show that $\tan\left(\frac{5\pi}{12}\right) = \sqrt{3} + 2$.

b Given $u = 1 + (\sqrt{3} + 2)i$, find iu and hence find $\text{Arg}(-\sqrt{3} - 2 + i)$.

12 Show that $\tan\left(\frac{11\pi}{12}\right) = \sqrt{3} - 2$ and hence find $\text{Arg}(1 + (\sqrt{3} - 2)i)$.

13 Find all values of n such that $(1 + \sqrt{3}\,i)^n - (1 - \sqrt{3}\,i)^n = 0$.

14 Find all values of n such that $(1 + \sqrt{3}\,i)^n + (1 - \sqrt{3}\,i)^n = 0$.

CONSOLIDATE

15 Express each of the following in polar form, giving angles in degrees and min

a $3 + 4i$

b $7 - 24i$

c $-5 + 12i$

d $-4 - 4i$

16 If $u = 6\,\text{cis}(12°)$ and $v = 3\,\text{cis}(23°)$, find each of the following in polar form, giving angles in degrees.

a uv

b $\frac{v}{u}$

c u^2

d v^3

e u^5v^4

f $\frac{v^6}{u^3}$

17 Given $u = 3 + 2i$ and $v = 7\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$, find each of the following, express your answers in exact rectangular form.

a uv

b $2u - 3v$

c $\frac{v}{u}$

d v^2

18 a If $z = 2 + 2i$, find each of the following.

i z^8

ii $\text{Arg}(z^8)$

b If $z = -3\sqrt{3} + 3i$, find each of the following.

i z^6

ii $\text{Arg}(z^6)$

c If $z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$, find each of the following.

i z^9

ii $\text{Arg}(z^9)$

d If $z = 2\sqrt{3} - 2i$, find each of the following.

i z^7

ii $\text{Arg}(z^7)$

19 Let $u = \frac{1}{2}(\sqrt{3} - i)$.

a Find $\overline{u}$, $\frac{1}{u}$ and u^6, giving all answers in rectangular form.

b Find $\text{Arg}(\overline{u})$, $\text{Arg}\left(\frac{1}{u}\right)$ and $\text{Arg}(u^6)$.

c Is $\text{Arg}(\overline{u})$ equal to $-\text{Arg}(u)$?

d Is $\text{Arg}\left(\frac{1}{u}\right)$ equal to $-\text{Arg}(u)$?

e Is $\text{Arg}(u^6)$ equal to $6\,\text{Arg}(u)$?

20 a Let $u = -1 + \sqrt{3}i$ and $v = -2 - 2i$.

i Find $\text{Arg}(u)$.

ii Find $\text{Arg}(v)$.

iii Find $\text{Arg}(uv)$.

iv Find $\text{Arg}\left(\frac{u}{v}\right)$.

v Is $\text{Arg}(uv)$ equal to $\text{Arg}(u) + \text{Arg}(v)$?

vi Is $\text{Arg}\left(\frac{u}{v}\right)$ equal to $\text{Arg}(u) - \text{Arg}(v)$?

b Let $u = -\sqrt{3} + i$ and $v = -3 + 3i$.

i Find $\text{Arg}(u)$.

ii Find $\text{Arg}(v)$.

iii Find $\text{Arg}(uv)$.

iv Find $\text{Arg}\left(\frac{u}{v}\right)$.

v Is $\text{Arg}(uv)$ equal to $\text{Arg}(u) + \text{Arg}(v)$?

vi Is $\text{Arg}\left(\frac{u}{v}\right)$ equal to $\text{Arg}(u) - \text{Arg}(v)$?

21 a Let $u = \frac{1}{4}(\sqrt{3} - i)$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$.

i Find uv, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find uv, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\sin\left(\frac{\pi}{12}\right)$.

iv Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{\pi}{12}\right)$.

b Let $u = \sqrt{2}(1 - i)$ and $v = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$.

i Find uv, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find uv, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

iv Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{5\pi}{12}\right)$.

22 a Let $u = -4 - 4\sqrt{3}\,i$ and $v = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$.

i Find $\frac{u}{v}$, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find $\frac{u}{v}$, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\cos\left(\frac{\pi}{12}\right)$.

iv Using the formula $\cos(x - y)$, verify your exact value for $\cos\left(\frac{\pi}{12}\right)$.

b Let $u = -1 - \sqrt{3}i$ and $v = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$.

i Find $\frac{u}{v}$, working with both numbers in Cartesian form and giving your answer in Cartesian form.

ii Find $\frac{u}{v}$, working with both numbers in polar form and giving your answer in polar form.

iii Hence, deduce the exact value of $\cos\left(\frac{7\pi}{12}\right)$.

iv Using the formula $\cos(x - y)$, verify your exact value for $\cos\left(\frac{7\pi}{12}\right)$.

23 a i Show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

ii Let $u = 1 + (\sqrt{2} - 1)i$ and hence find $\text{Arg}(u)$.

iii Find iu and hence find $\text{Arg}((1 - \sqrt{2}) + i)$.

b i Show that $\tan\left(\frac{7\pi}{12}\right) = -(\sqrt{3} + 2)$.

ii Hence, find $\text{Arg}(-1 + (\sqrt{3} + 2)i)$.

iii Hence, find $\text{Arg}(1 - (\sqrt{3} + 2)i)$.

iv Hence, find $\text{Arg}((\sqrt{3} + 2) + i)$.

24 Find all values of n such that:

a $(1 + i)^n + (1 - i)^n = 0$

b $(1 + i)^n - (1 - i)^n = 0$

c $(\sqrt{3} + i)^n - (\sqrt{3} - i)^n = 0$

d $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 0$.

MASTER

25 If $z = \text{cis}(\theta)$, show that:

a $|z + 1| = 2\cos\left(\frac{\theta}{2}\right)$

b $\text{Arg}(1 + z) = \frac{\theta}{2}$

c $\frac{1 + z}{1 - z} = i\cot\left(\frac{\theta}{2}\right)$.

26 Use de Moivre's theorem to show that:

a i $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

ii $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b i $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

ii $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$.

3.4 Polynomial equations

Quadratic equations

Recall the quadratic equation $az^2 + bz + c = 0$. If the coefficients a, b and c are all real, then the roots depend upon the discriminant, $\Delta = b^2 - 4ac$.

If $\Delta > 0$, then there are two distinct real roots.

If $\Delta = 0$, then there is one real root.

If $\Delta < 0$, then there is one pair of complex conjugate roots.

studyon

Factorisation of polynomials over C

Concept summary

Practice questions

Relationship between the roots and coefficients

Given a quadratic equation with real coefficients, if the discriminant is negative, the roots occur in complex conjugate pairs. A relationship can be formed between the roots and the coefficients.

$z = \dfrac{-(i-8) \pm \sqrt{-13+84i}}{2}$

$= \dfrac{8-i \pm (6+7i)}{2}$

$= 7+3i, 1-4i$

32 a i $4 = 1+i$

$4^2 = (1+i)^2$

$= 1+2i+i^2$

$= 2i$

ii $4^3 = (1+i)^3$

$= 1+3i+3i^2+i^3$

$= 1+3i-3-i$

$= -2+2i$

iii $4^4 = \left(1^2\right)^2$

$= (2i)^2$

$= 4i^2$

$= -4$

b i $4 = 3-4i$

$4^2 = (3-4i)^2$

$= 9-24i+16i^2$

$= -7-24i$

ii $4^3 = (3-4i)^3 = 4^2 \times 4$

$= -(7+24i)(3-4i)$

$= -\left(21+72i-28i-96i^2\right)$

$= -117-44i$

iii $4^4 = \left(4^2\right)^2$

$= \left(-(7+24i)\right)^2$

$= 49+336i+576i^2$

$= -527+336i$

c $4 = a+bi$

i $4^2 = (a+bi)^2$

$= a^2+2abi+b^2i^2$

$= a^2-b^2+2abi$

ii $4^3 = (a+bi)^3$

Use binomial theorem expansion

$= a^3+3a^2bi+3ab^2i^2+b^3i^3$

$= a^3-3ab^2+\left(3a^2b-b^3\right)i$

iii $4^4 = (a+bi)^4$

$= a^4+4a^3bi+6a^2b^2i^2+4ab^3i^3+b^4i^4$

$= a^4-6a^2b^2+b^4+\left(4a^3b-4ab^3\right)i$

Exercise 3.3 — Complex numbers in polar form

1 a $z = 1+\sqrt{3}i$

First quadrant

$x = 1, y = \sqrt{3}$

$|z| = \sqrt{x^2+y^2}$

$= \sqrt{1+3}$

$= 2$

$\tan(\theta) = \dfrac{y}{x}$

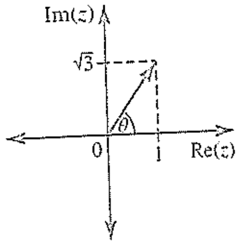
$= \sqrt{3}$

$\theta = \text{Arg}(z)$

$= \tan^{-1}\left(\sqrt{3}\right)$

$= \dfrac{\pi}{3}$

$z = 2\text{ cis}\left(\dfrac{\pi}{3}\right)$



b $z = -1+i$

Second quadrant

$|z| = \sqrt{x^2+y^2}$

$= \sqrt{(-1)^2+(1)^2}$

$= \sqrt{2}$

$x = -1, y = 1$

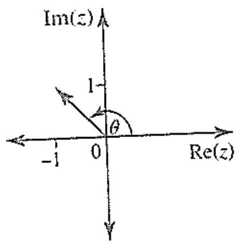
$\theta = \text{Arg}(z)$

$= \tan^{-1}(-1)+\pi$

$= -\dfrac{\pi}{4}+\pi$

$= \dfrac{3\pi}{4}$

$z = \sqrt{2}\text{ cis}\left(\dfrac{3\pi}{4}\right)$



c $z = -2-2\sqrt{3}i$

Third quadrant

$x = -2, y = -2\sqrt{3}$

$|z| = \sqrt{x^2+y^2}$

$= \sqrt{(-2)^2+\left(-2\sqrt{3}\right)^2}$

$= \sqrt{4+12}$

$= \sqrt{16}$

$= 4$

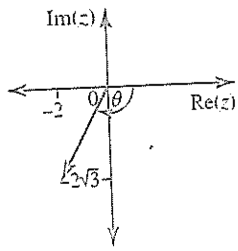
$\theta = \text{Arg}(z)$

$= -\pi+\tan^{-1}\left(\sqrt{3}\right)$

$= -\pi+\dfrac{\pi}{3}$

$= -\dfrac{2\pi}{3}$

$z = 4\text{ cis}\left(-\dfrac{2\pi}{3}\right)$



d $z=\sqrt{3}-i$

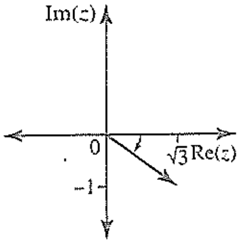
Fourth quadrant

$x=\sqrt{3}, y=-1$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{3+1} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \operatorname{Arg}(z) \\
&= \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) \\
&= -\frac{\pi}{6}
\end{aligned}
$$

$z=2\operatorname{cis}\left(-\frac{\pi}{6}\right)$



e $z=4$

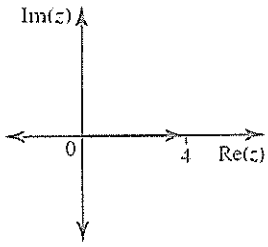
$x=4$

$y=0$

$|z|=4$

$$
\begin{aligned}
\theta &= \operatorname{Arg}(z) \\
&= 0
\end{aligned}
$$

$z=4\operatorname{cis}(0)$



f $z=-2i$

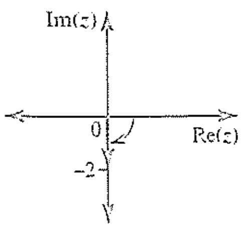
$x=0$

$y=-2$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{0+4} \\
&= 2
\end{aligned}
$$

$\theta=-\frac{\pi}{2}$

$z=2\operatorname{cis}\left(-\frac{\pi}{2}\right)$



2 a $z=\sqrt{3}+i$

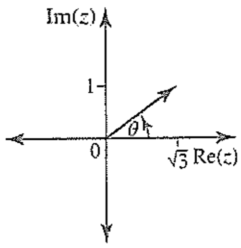
First quadrant

$x=\sqrt{3},\ y=1$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{3+1} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \operatorname{Arg}(z) \\
&= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
&= \frac{\pi}{6}
\end{aligned}
$$

$z=2\operatorname{cis}\left(\frac{\pi}{6}\right)$



b $z=-1+\sqrt{3}i$

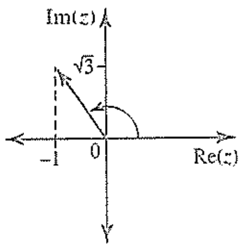
Second quadrant

$x=-1, y=\sqrt{3}$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{1+3} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \operatorname{Arg}(z) \\
&= \pi-\tan^{-1}\left(\sqrt{3}\right) \\
&= \pi-\frac{\pi}{3} \\
&= \frac{2\pi}{3}
\end{aligned}
$$

$z=2\operatorname{cis}\left(\frac{2\pi}{3}\right)$



c $z=-\sqrt{3}-i$

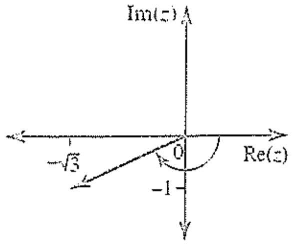
Third quadrant

$x=-\sqrt{3}, y=-1$

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{3+1} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \operatorname{Arg}(z) \\
&= -\pi+\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
&= -\pi+\frac{\pi}{6} \\
&= -\frac{5\pi}{6}
\end{aligned}
$$

$z=2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$



d $z = 2 - 2i$

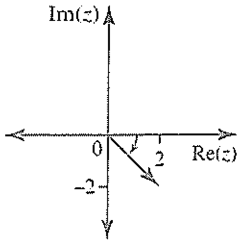
Fourth quadrant

$x = 2, y = -2$

$$\begin{aligned}|z| &= \sqrt{x^2 + y^2}\\ &= \sqrt{4+4}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\theta &= \text{Arg}(z)\\ &= \tan^{-1}\left(-\frac{\pi}{4}\right)\end{aligned}$$

$$z = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

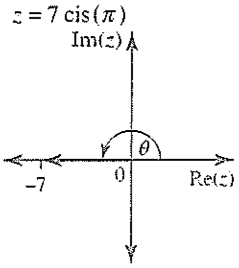


e $z = -7$

$x = -7, y = 0$

$|z| = 7, \theta = \pi$

$z = 7\,\text{cis}(\pi)$

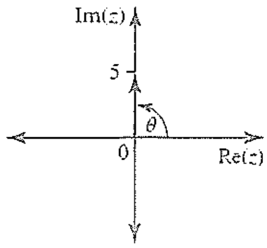


f $z = 5i$

$x = 0, y = 5$

$|z| = 5, \theta = \frac{\pi}{2}$

$z = 5\,\text{cis}\left(\frac{\pi}{2}\right)$



3 a
$$\begin{aligned}4\,\text{cis}\left(-\frac{\pi}{3}\right) &= 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\\ &= 4\cos\left(\frac{\pi}{3}\right) - 4i\sin\left(\frac{\pi}{3}\right)\\ &= 4 \times \frac{1}{2} - 4i \times \frac{\sqrt{3}}{2}\\ &= 2 - 2\sqrt{3}i\end{aligned}$$

b
$$\begin{aligned}8\,\text{cis}\left(-\frac{\pi}{2}\right) &= 8\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)\\ &= 8 \times 0 + 8 \times i \times -1\\ &= -8i\end{aligned}$$

4 a
$$\begin{aligned}6\sqrt{2}\,\text{cis}(-135°) &= 6\sqrt{2}\left(\cos(-135°) + i\sin(-135°)\right)\\ &= 6\sqrt{2}\cos(135°) - 6\sqrt{2}i\sin(135°)\\ &= 6\sqrt{2} \times -\frac{1}{\sqrt{2}} - 6\sqrt{2} \times \frac{1}{\sqrt{2}}i\\ &= -6 - 6i\end{aligned}$$

b
$$\begin{aligned}5\,\text{cis}(126°52') &= 5\left(\cos(126°52') + i\sin(126°52')\right)\\ &= 5 \times -\frac{3}{5} + i \times 5 \times \frac{4}{5}\\ &= -3 + 4i\end{aligned}$$

5 $u = 6\,\text{cis}\left(-\frac{\pi}{3}\right)$

$\bar{u} = 6\,\text{cis}\left(\frac{\pi}{3}\right)$

$$\begin{aligned}\bar{u}^{-1} &= \frac{1}{6\,\text{cis}\left(\frac{\pi}{3}\right)}\\ &= \frac{1}{6}\,\text{cis}\left(-\frac{\pi}{3}\right)\\ &= \frac{1}{6}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\\ &= \frac{1}{6}\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right)\\ &= \frac{1}{6}\left(\frac{1}{2} - i \times \frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{12} - \frac{\sqrt{3}}{12}i\end{aligned}$$

6 $u = \frac{\sqrt{2}}{4}\,\text{cis}\left(\frac{3\pi}{4}\right)$

$\bar{u} = \frac{\sqrt{2}}{4}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$$\begin{aligned}\bar{u}^{-1} &= \frac{4}{\sqrt{2}}\,\text{cis}\left(\frac{3\pi}{4}\right)\\ &= \frac{4}{\sqrt{2}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\\ &= \frac{4}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\\ &= -2 + 2i\end{aligned}$$

7 $u = 4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

a
$$\begin{aligned}uv &= \left(4\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)\right)\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)\\ &= 4\sqrt{2} \times \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4} + \frac{\pi}{4}\right)\\ &= 8\,\text{cis}\left(-\frac{\pi}{2}\right)\\ &= -8i\end{aligned}$$

b $\frac{u}{v}=\frac{4\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)}{\left(\sqrt{2}\ \text{cis}\right)\left(\frac{\pi}{4}\right)}$

$=4\ \text{cis}\left(-\frac{3\pi}{4}-\frac{\pi}{4}\right)$

$=4\ \text{cis}(-\pi)$

$=-4$

8 $u=4\ \text{cis}\left(\frac{\pi}{3}\right)$

$v=\frac{1}{2}\ \text{cis}\left(-\frac{2\pi}{3}\right)$

a $uv=\left(4\ \text{cis}\left(\frac{\pi}{3}\right)\left(\frac{1}{2}\ \text{cis}\right)\left(-\frac{2\pi}{3}\right)\right)$

$=4\times\frac{1}{2}\ \text{cis}\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)$

$=2\ \text{cis}\left(-\frac{\pi}{3}\right)$

$=2\cos\left(-\frac{\pi}{3}\right)+2i\sin\left(-\frac{\pi}{3}\right)$

$=1-\sqrt{3}i$

b $\frac{u}{v}=\frac{4\ \text{cis}\left(\frac{\pi}{3}\right)}{\frac{1}{2}\ \text{cis}\left(-\frac{2\pi}{3}\right)}$

$=4\times 2\ \text{cis}\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)$

$=8\ \text{cis}(\pi)$

$=-8$

9 $u=-1-i$

(c) $x=-1, y=-1$

$|z|=\sqrt{2}$

$\theta=\text{Arg}(z)$

$=-\frac{3\pi}{4}$

$u=\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)$

$\arg\left(u^{10}\right)=-\frac{3\pi}{4}\times 10$

$=-\frac{15\pi}{2}$

$\text{Arg}\left(u^{10}\right)=-\frac{15\pi}{2}+8\pi$ Correcting for principle argument.

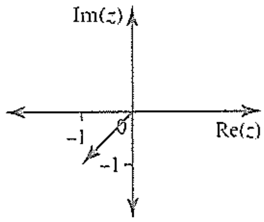
$=\frac{\pi}{2}$

$u^{10}=\left(\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)\right)^{10}$

$=\left(\sqrt{2}\right)^{10}\ \text{cis}\left(\frac{\pi}{2}\right)$

$=2^5\ \text{cis}\left(\frac{\pi}{2}\right)$

$=32i$



10 $\frac{(-1+i)^6}{\left(\sqrt{3}-i\right)^4}=\frac{\left(\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)\right)^6}{\left(2\ \text{cis}\left(-\frac{\pi}{6}\right)\right)^4}$

$=\frac{\left(\sqrt{2}\right)^6}{2^4}\ \text{cis}\left(-\frac{6\pi}{4}+\frac{4\times\pi}{6}\right)$

$=\frac{1}{2}\ \text{cis}\left(-\frac{5\pi}{6}\right)$

$=\frac{1}{2}\left(\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)\right)$

$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}-i\times\frac{1}{2}\right)$

$=-\frac{\sqrt{3}}{4}-\frac{1}{4}i$

11 a $\tan\left(\frac{5\pi}{12}\right)=\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{6}\right)}{1-\tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)}$

$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$

$=\frac{1+\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}}$

$=\frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}$

$=\frac{3+\sqrt{3}}{3-\sqrt{3}}\times\frac{3+\sqrt{3}}{3+\sqrt{3}}$

$=\frac{9+6\sqrt{3}+3}{9-3}$

$=\frac{12+6\sqrt{3}}{6}$

$=2+\sqrt{3}$

b $u=1+\left(\sqrt{3}+2\right)i$

$\text{Arg}(u)=\tan^{-1}\left(2+\sqrt{3}\right)$

$=\frac{5\pi}{12}$

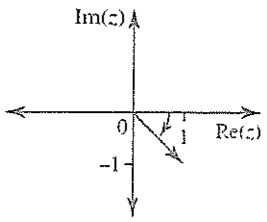
$$iu = \left(1+\left(\sqrt{3}+2\right)i\right)i$$
$$= i+\left(\sqrt{3}+2\right)i^2$$
$$= -\sqrt{3}-2+i$$
$$\text{Arg}(iu) = \frac{5\pi}{12}+\frac{\pi}{2}$$
$$= \frac{11\pi}{12}$$

12 $\tan\left(\frac{11\pi}{12}\right) = \tan\left(\frac{3\pi}{4}+\frac{\pi}{6}\right)$

$$= \frac{\tan\left(\frac{3\pi}{4}\right)+\tan\left(\frac{\pi}{6}\right)}{1-\tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$
$$= \frac{-1+\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}$$
$$= \frac{-3+\sqrt{3}}{3+\sqrt{3}}\times\frac{3-\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{-9+6\sqrt{3}-3}{9-3}$$
$$= \frac{6\sqrt{3}-12}{6}$$
$$= \sqrt{3}-2$$

$u = 1+\left(\sqrt{3}-2\right)i$, in fourth quadrant

$$\text{Arg}(u) = \tan^{-1}\left(\sqrt{3}-2\right)$$
$$= \frac{11\pi}{12}-\pi$$
$$= -\frac{\pi}{12}$$



13 $0 = \left(1+\sqrt{3}i\right)^n-\left(1-\sqrt{3}i\right)^n$

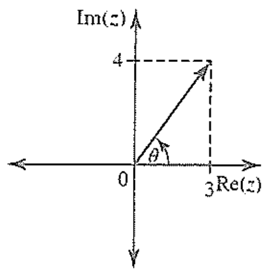
$$0 = \left(2\,\text{cis}\left(\frac{\pi}{3}\right)\right)^n-\left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^n$$
$$0 = 2^n\,\text{cis}\left(\frac{n\pi}{3}\right)-2^n\,\text{cis}\left(-\frac{n\pi}{3}\right)$$
$$0 = 2^n\left(\cos\left(\frac{n\pi}{3}\right)+i\sin\left(\frac{n\pi}{3}\right)-\left(\cos\left(\frac{n\pi}{3}\right)-i\sin\left(\frac{n\pi}{3}\right)\right)\right)$$
$$0 = 2\times2^n i\sin\left(\frac{n\pi}{3}\right)$$
$$0 = \sin\left(\frac{n\pi}{3}\right)$$
$$\frac{n\pi}{3} = k\pi$$
$$n = 3k, k\in Z$$

14 $0 = \left(1+\sqrt{3}i\right)^n+\left(1-\sqrt{3}i\right)$

$$0 = \left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)^n+\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^n$$
$$0 = 2^n\,\text{cis}\left(\frac{n\pi}{6}\right)+2^n\,\text{cis}\left(-\frac{n\pi}{6}\right)$$
$$0 = 2^n\left(\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)+\left(\cos\left(\frac{n\pi}{6}\right)-i\sin\left(\frac{n\pi}{6}\right)\right)\right)$$
$$0 = 2\times2^n\cos\left(\frac{n\pi}{6}\right)$$
$$0 = \cos\left(\frac{n\pi}{6}\right)$$
$$\frac{n\pi}{6} = (2k+1)\frac{\pi}{2}$$
$$n = \frac{3}{2}(2k+1)$$
$$= 3k+\frac{3}{2} \qquad k\in Z$$

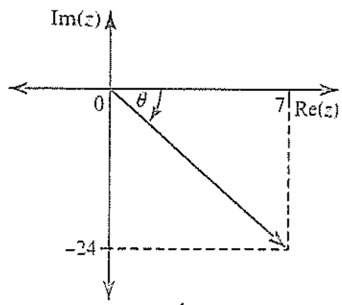
15 a $z = 3+4i$

$$x = 3 \quad y = 4$$
$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{9+16}$$
$$= 5$$
$$\theta = \text{Arg}(z)$$
$$= \tan^{-1}\left(\frac{4}{3}\right)$$
$$= 53°8'$$
$$z = 5\,\text{cis}(53°8')$$



b $z = 7-24i$

$$x = 7 \quad y = -24$$
$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{(7)^2+(-24)^2}$$
$$= \sqrt{625}$$
$$= 25$$
$$\theta = \text{Arg}(z)$$
$$= \tan^{-1}\left(-\frac{24}{7}\right)$$
$$= -73°44'$$
$$z = 25\,\text{cis}(-73°44')$$



c $z=-5+12i$

$x=-5 \quad y=12$

$r=\sqrt{x^2+y^2}$

$=\sqrt{(-5)^2+(12)^2}$

$=\sqrt{169}$

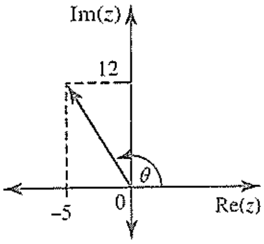
$=13$

$\theta=\text{Arg}(z)$

$=\tan^{-1}\left(\frac{12}{-5}\right)$

$=-67.38$

$z=13\,\text{cis}(112°37')$



d $z=-4-4i$

$x=-4 \quad y=-4$

$r=\sqrt{x^2+y^2}$

$=\sqrt{(-4)^2+(-4)^2}$

$=\sqrt{32}$

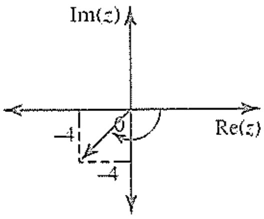
$=4\sqrt{2}$

$\theta=\text{Arg}(z)$

$=180-\tan^{-1}(1)$

$=-135°$

$z=4\sqrt{2}\,\text{cis}(-135°)$



16 $u=6\,\text{cis}(12°) \qquad v=3\,\text{cis}(23°)$

a $uv=6\,\text{cis}(12°)\times 3\,\text{cis}(23°)$

$=6\times 3\,\text{cis}(12+23)$

$=18\,\text{cis}(35°)$

b $\frac{v}{u}=\frac{3\,\text{cis}(23°)}{6\,\text{cis}(12°)}$

$=\frac{3}{6}\,\text{cis}(23°-12°)$

$=\frac{1}{2}\,\text{cis}(11°)$

c $u^2=(6\,\text{cis}(12°))^2$

$=6^2\,\text{cis}(2\times 12°)$

$=36\,\text{cis}(24°)$

d $v^3=(3\,\text{cis}(23°))^3$

$=3^3\,\text{cis}(3\times 23°)$

$=27\,\text{cis}(69°)$

e $u^5v^4=(6\,\text{cis}(12°))^5\times(3\,\text{cis}(23°))^4$

$=6^5\times 3^4\,\text{cis}(12\times 5+23\times 4)$

$=629\,856\,\text{cis}(152°)$

f $\frac{v^6}{u^3}=\frac{(3\,\text{cis}(23°))^6}{(6\,\text{cis}(12°))^3}$

$=\frac{3^6}{6^3}\,\text{cis}(6\times 23°-12°\times 3)$

$=\frac{27}{8}\,\text{cis}(102°)$

17 $u=3+2i$

$v=7\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

$=7\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$

$=7\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

$=-7+7i$

a $uv=(3+2i)(-7+7i)$

$=-21-14i+21i+14i^2$

$=-35+7i$

b $2u-3v=2(3+2i)-3(-7+7i)$

$=6+4i-(-21+21i)$

$=27-17i$

c $\frac{v}{u}=\frac{-7+7i}{3+2i}\times\frac{3-2i}{3-2i}$

$=\frac{-21+21i+14i-14i^2}{9-4i^2}$

$=-\frac{7}{13}+\frac{35}{13}i$

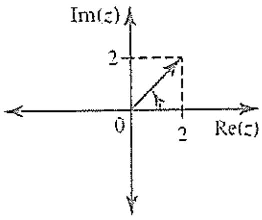
d $v^2=(-7+7i)^2$

$=49-98i+49i^2$

$=-98i$

18 a $z=2+2i$

$=2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$



i $z^8=(2\sqrt{2})^8\,\text{cis}\left(8\times\frac{\pi}{4}\right)$

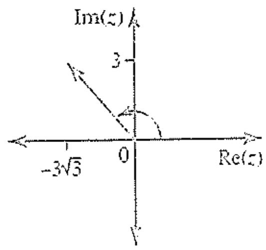
$=4096\,\text{cis}(2\pi)$

$=4096\,\text{cis}(0)$

$=4096$

ii $\text{Arg}(z^8)=0$

b



$z=-3\sqrt{3}+3i$

$x=-3\sqrt{3} \qquad y=3$

$r=\sqrt{\left(-3\sqrt{3}\right)^2+(3)^2}$

$=6$

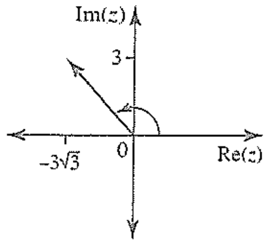
$$\text{Arg}(z) = \pi - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$= \pi - \frac{\pi}{6}$$
$$= \frac{5\pi}{6}$$

$$z = -3\sqrt{3} + 3i$$
$$= 6\,\text{cis}\left(\frac{5\pi}{6}\right)$$

i $z^6 = 6^6\,\text{cis}\left(6 \times \frac{5\pi}{6}\right)$
$$= 6^6\,\text{cis}(5\pi)$$
$$= 46656\,\text{cis}(5\pi - 4\pi)$$
$$= 46656\,\text{cis}(\pi)$$
$$= -46656$$

ii $\text{Arg}(z^6) = \pi$

c



$$z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$$
$x = -\frac{5}{2}$ $\quad y = -\frac{5\sqrt{3}}{2}$
$$r = \sqrt{\left(-\frac{5}{2}\right)^2 + \left(-\frac{5\sqrt{3}}{2}\right)^2}$$
$$= 5$$

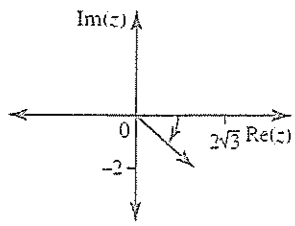
$$\text{Arg}(z) = -\pi + \tan^{-1}(\sqrt{3})$$
$$= -\pi + \frac{\pi}{3}$$
$$= -\frac{2\pi}{3}$$

$$z = 5\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

i $z^9 = 5^9\,\text{cis}\left(9 \times -\frac{2\pi}{3}\right)$
$$= 5^9\,\text{cis}(-4\pi)$$
$$= 1953125\,\text{cis}(0)$$
$$= 1953125$$

ii $\text{Arg}(z^9) = 0$

d



$$z = 2\sqrt{3} - 2i$$
$x = 2\sqrt{3}$ $\quad y = -2$
$$r = \sqrt{(2\sqrt{3})^2 + (-2)^2}$$
$$= 5$$

$$\text{Arg}(z) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$= -\frac{\pi}{6}$$
$$z = 4\,\text{cis}\left(-\frac{\pi}{6}\right)$$

i $z^7 = 4^7\,\text{cis}\left(-\frac{7\pi}{6}\right)$
$$= 4^7\,\text{cis}\left(-\frac{7\pi}{6} + 2\pi\right)$$
$$= 16384\,\text{cis}\left(\frac{5\pi}{6}\right)$$
$$= 16384\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$
$$= 16384\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -8192\sqrt{3} + 8192i$$

ii $\text{Arg}(z^7) = \frac{5\pi}{6}$

19 $u = \frac{1}{2}(\sqrt{3} - i)$
$$= \text{cis}\left(-\frac{\pi}{6}\right)$$

a $\bar{u} = \text{cis}\left(\frac{\pi}{6}\right)$
$$= \frac{1}{2}(\sqrt{3} + i)$$
$$\left(\frac{1}{u}\right) = \text{cis}\left(\frac{\pi}{6}\right)$$
$$= \frac{1}{2}(\sqrt{3} + i)$$
$$u^6 = \text{cis}(-\pi)$$
$$= \text{cis}(\pi)$$
$$= -1$$

b $\text{Arg}(\bar{u}) = \frac{\pi}{6}$

$\text{Arg}\left(\frac{1}{u}\right) = \frac{\pi}{6}$

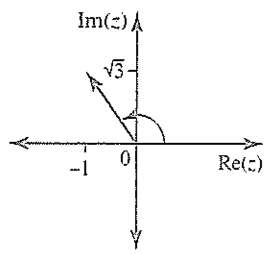
$\text{Arg}(u^6) = \pi$

c $\text{Arg}(\bar{u}) = -\text{Arg}(u)$ Yes

d $\text{Arg}\left(\frac{1}{u}\right) = -\text{Arg}(u)$ Yes

e $\text{Arg}(u^6) = 6\text{Arg}(u)$ No

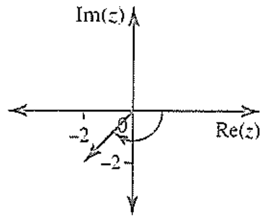
20 a i



$$u = -1 + \sqrt{3}i$$
$x = -1$ $\quad y = \sqrt{3}$
$$r = \sqrt{1 + 3}$$
$$= 2$$

$$\theta = \text{Arg}(u)$$
$$= \pi - \tan^{-1}(\sqrt{3})$$
$$= \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$
$$u = -1 + \sqrt{3}i$$
$$= 2\,\text{cis}\left(\frac{2\pi}{3}\right)$$

ii



$$v = -2 - 2i$$
$x = -2$ $\quad y = -2$
$$r = \sqrt{4 + 4}$$
$$= 2\sqrt{2}$$
$$\theta = \text{Arg}(v)$$
$$= -\pi - \tan^{-1}(1)$$
$$= -\pi + \frac{\pi}{4}$$
$$= -\frac{3\pi}{4}$$
$$v = -2 - 2i$$
$$= 2\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$$

iii $uv = 2\,\text{cis}\left(\frac{2\pi}{3}\right) \times 2\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$
$$= 4\sqrt{2}\,\text{cis}\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right)$$
$$= 4\sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$$
$$\Rightarrow \text{Arg}(uv) = -\frac{\pi}{12}.$$

iv $\frac{u}{v} = \frac{2\,\text{cis}\left(\frac{2\pi}{3}\right)}{2\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)}$
$$= \frac{1}{\sqrt{2}}\,\text{cis}\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}\,\text{cis}\left(\frac{17\pi}{12} - 2\pi\right)$$
$$= \frac{\sqrt{2}}{2}\,\text{cis}\left(-\frac{7\pi}{12}\right)$$
$$\text{Arg}\left(\frac{u}{v}\right) = -\frac{7\pi}{12}$$
so $\text{Arg}(u) = \frac{2\pi}{3}$, $\text{Arg}(v) = -\frac{3\pi}{4}$

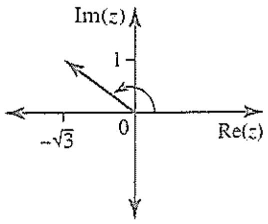
v $\text{Arg}(uv) = \text{Arg}(u) + \text{Arg}(v)$

In this case, yes, but not in general

vi $\text{Arg}\left(\frac{u}{v}\right) = -\frac{7\pi}{12} \neq \text{Arg}(u) - \text{Arg}(v)$

No

b **i**



$u=-\sqrt{3}+i$

$x=-\sqrt{3} \qquad y=1$

$r=\sqrt{3+1}$

$=2$

$\theta=\text{Arg}(u)$

$=\pi-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

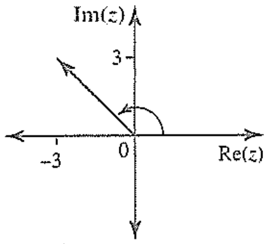
$=\pi-\frac{\pi}{6}$

$=\frac{5\pi}{6}$

$u=-\sqrt{3}+i$

$=2\text{ cis}\left(\frac{5\pi}{6}\right)$

ii



$v=-3+3i$

$x=-3 \qquad y=3$

$r=\sqrt{9+9}$

$=3\sqrt{2}$

$\theta=\text{Arg}(v)$

$=\pi-\tan^{-1}(1)$

$=\frac{3\pi}{4}$

$v=-\sqrt{3}+i$

$=3\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$

iii $uv=2\text{ cis}\left(\frac{5\pi}{6}\right)\times 3\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$

$=6\sqrt{2}\text{ cis}\left(\frac{5\pi}{6}+\frac{3\pi}{4}\right)$

$=6\sqrt{2}\text{ cis}\left(\frac{19\pi}{12}-2\pi\right)$

$=6\sqrt{2}\text{ cis}\left(-\frac{5\pi}{12}\right)$

iv $\frac{u}{v}=\frac{2\text{ cis}\left(\frac{5\pi}{6}\right)}{3\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)}$

$=\frac{2}{3\sqrt{2}}\text{ cis}\left(\frac{5\pi}{6}-\frac{3\pi}{4}\right)$

$=\frac{\sqrt{2}}{3}\text{ cis}\left(\frac{\pi}{12}\right)$

$\text{Arg}\left(\frac{u}{v}\right)=\frac{\pi}{12}$

so

$\text{Arg}(u)=\frac{5\pi}{6},\ \text{Arg}(v)=-\frac{3\pi}{4}$

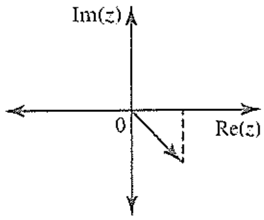
v $\text{Arg}(uv)=-\frac{5\pi}{12}\neq\text{Arg}(u)+\text{Arg}(v)$

No

vi $\text{Arg}\left(\frac{u}{v}\right)=\frac{\pi}{12}=\text{Arg}(u)-\text{Arg}(v)$

In this case, yes, but not in general

21 **a**



$u=\frac{1}{4}\left(\sqrt{3}-i\right)$

$x=\frac{\sqrt{3}}{4} \qquad y=-\frac{1}{4}$

$r=\sqrt{\frac{9}{16}+\frac{1}{4}}$

$=\frac{1}{2}$

$\theta=\text{Arg}(u)$

$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$=-\frac{\pi}{6}$

$u=\frac{1}{4}\left(\sqrt{3}-i\right)$

$=\frac{1}{2}\text{ cis}\left(-\frac{\pi}{6}\right)$

$v=\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$

$=1+i$

i $uv=\frac{1}{4}\left(\sqrt{3}-i\right)(1+i)$

$=\frac{1}{4}\left(\sqrt{3}-i\right)+\frac{1}{4}\left(\sqrt{3}i-i^2\right)$

$=\frac{1}{4}\left(\sqrt{3}+i\right)+\frac{1}{4}\left(\sqrt{3}-i\right)i$

ii $uv=\frac{1}{2}\text{ cis}\left(-\frac{\pi}{6}\right)\times\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$

$=\frac{\sqrt{2}}{2}\text{ cis}\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)$

$=\frac{\sqrt{2}}{2}\text{ cis}\left(\frac{\pi}{12}\right)$

iii $uv=\frac{\sqrt{2}}{2}\text{ cis}\left(\frac{\pi}{12}\right)+\frac{\sqrt{2}}{2}i\sin\left(\frac{\pi}{12}\right)$

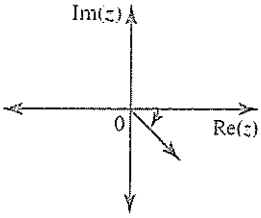
Equating imaginary parts

$\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{12}\right)=\frac{1}{4}\left(\sqrt{3}-1\right)$

$$\sin\left(\frac{\pi}{12}\right)=\frac{2}{\sqrt{2}}\times\frac{1}{4}\left(\sqrt{3}-1\right)\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)\quad\text{shown}$$

iv $$\sin\left(\frac{\pi}{12}\right)=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$
$$=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$
$$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}$$
$$=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

b



$$u=\sqrt{2}(1-i)$$
$$x=\sqrt{2}\qquad y=-\sqrt{2}$$
$$r=\sqrt{2+2}$$
$$=2$$
$$\theta=\operatorname{Arg}(u)$$
$$=\tan^{-1}(-1)$$
$$=-\frac{\pi}{4}$$
$$u=\sqrt{2}(1-i)$$
$$=2\operatorname{cis}\left(-\frac{\pi}{4}\right)$$
$$v=2\operatorname{cis}\left(\frac{2\pi}{3}\right)$$
$$=2\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$$
$$=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$=-1+\sqrt{3}i$$

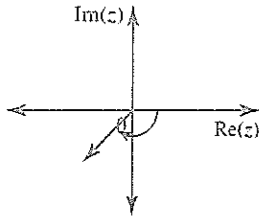
i $$uv=\left(2-\sqrt{2}i\right)\left(-1+\sqrt{3}i\right)$$
$$=-\sqrt{2}+\sqrt{2}i+\sqrt{6}i-\sqrt{6}i^2$$
$$=\left(\sqrt{6}-\sqrt{2}\right)+\left(\sqrt{6}+\sqrt{2}\right)i$$

ii $$uv=2\operatorname{cis}\left(-\frac{\pi}{4}\right)\times2\operatorname{cis}\left(\frac{2\pi}{3}\right)$$
$$=4\operatorname{cis}\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)$$
$$=4\operatorname{cis}\left(\frac{5\pi}{12}\right)$$

iii $$uv=\cos\left(\frac{5\pi}{12}\right)+4i\sin\left(\frac{5\pi}{12}\right)$$
Equating imaginary parts
$$4\sin\left(\frac{5\pi}{12}\right)=\sqrt{6}+\sqrt{2}$$
$$\sin\left(\frac{5\pi}{12}\right)=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

iv $$\sin\left(\frac{5\pi}{12}\right)=\sin\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)$$
$$=\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$
$$=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{\sqrt{2}}{2}$$
$$=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

22



$$u=-4-4\sqrt{3}i$$
$$x=-4\qquad y=-4\sqrt{3}$$
$$r=\sqrt{16+48}$$
$$=8$$
$$\theta=\operatorname{Arg}(u)$$
$$=-\pi+\tan^{-1}\left(\sqrt{3}\right)$$
$$=-\pi+\frac{\pi}{3}$$
$$=-\frac{2\pi}{3}$$
$$u=-4-4\sqrt{3}i$$
$$=8\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$
$$v=\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$
$$=-1-i$$

a i $$\frac{u}{v}=\frac{-4-4\sqrt{3}i}{-1-i}=\frac{4\left(1+\sqrt{3}i\right)}{1+i}\times\frac{1-i}{1-i}$$
$$=\frac{4\left(1+\sqrt{3}i-i\sqrt{3}i^2\right)}{1-i^2}$$
$$=2\left(\sqrt{3}+1\right)+2\left(\sqrt{3}-1\right)i$$

ii $$\frac{u}{v}=\frac{8\operatorname{cis}\left(-\frac{2\pi}{3}\right)}{\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)}$$
$$=\frac{8}{\sqrt{2}}\operatorname{cis}\left(-\frac{2\pi}{3}+\frac{3\pi}{4}\right)\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=4\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)$$

iii $$\frac{u}{v}=4\sqrt{2}\cos\left(\frac{\pi}{12}\right)+i4\sqrt{2}\sin\left(\frac{\pi}{12}\right)$$
Equating real parts
$$4\sqrt{2}\cos\left(\frac{\pi}{12}\right)=2\left(\sqrt{3}+1\right)$$
$$\cos\left(\frac{\pi}{12}\right)=\frac{2}{4\sqrt{2}}\left(\sqrt{3}+1\right)\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\quad\text{shown}$$

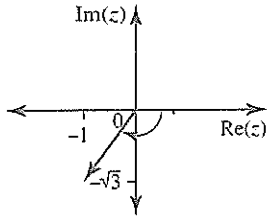
iv $\cos\left(\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$

$=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}$

$=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

b



$u=-1-\sqrt{3}i$

$x=-1 \qquad y=-\sqrt{3}$

$r=\sqrt{1+3}$

$=2$

$\theta=\operatorname{Arg}(u)$

$=-\pi+\tan^{-1}\left(\sqrt{3}\right)$

$=-\pi+\frac{\pi}{3}$

$=-\frac{2\pi}{3}$

$u=-1-\sqrt{3}i$

$=2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

$v=\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

$=\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$

$=\sqrt{2\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)}$

$=-1+i$

$=\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

i $\frac{u}{v}=\frac{-1-\sqrt{3}i}{-1+i}=\frac{1+\sqrt{3}i}{1-i}\times\frac{1+i}{1+i}$

$=\frac{1+\sqrt{3}i+i+\sqrt{3}i^2}{1-i^2}$

$=\frac{1}{2}\left(1-\sqrt{3}\right)+\frac{1}{2}\left(\sqrt{3}+1\right)i$

ii $\frac{u}{v}=\frac{2\operatorname{cis}\left(-\frac{2\pi}{3}\right)}{\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}$

$=\frac{2}{\sqrt{2}}\operatorname{cis}\left(-\frac{2\pi}{3}-\frac{3\pi}{4}\right)$

$=\sqrt{2}\operatorname{cis}\left(-\frac{17\pi}{12}+2\pi\right)$

$=\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

iii $\frac{u}{v}=\sqrt{2}\cos\left(\frac{7\pi}{12}\right)+i\sqrt{2}\sin\left(\frac{7\pi}{12}\right)$

Equating real parts

$\sqrt{2}\cos\left(\frac{7\pi}{12}\right)=\frac{1}{2}\left(1-\sqrt{3}\right)$

$\cos\left(\frac{7\pi}{12}\right)=\frac{1}{2}\left(1-\sqrt{3}\right)\times\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$ shown

iv $\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)$

$=\cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$=\frac{-\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}$

$=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

23 a i $\tan(2A)=\frac{2\tan(A)}{1-\tan^2(A)}$

$A=\frac{\pi}{8} \qquad 2A=\frac{\pi}{4} \qquad$ let $a=\tan\left(\frac{\pi}{8}\right)$

$\frac{2a}{1-a^2}=1$

$1-a^2=2a$

$a^2+2a=1$

$a^2+2a+1=2$

$(a+1)^2=2$

$a+1=\pm\sqrt{2}$ but $a=\tan\left(\frac{\pi}{8}\right)>0$, take positive

$a=\tan\left(\frac{\pi}{8}\right)$

$=\sqrt{2}-1$ shown

ii $u=1+\left(\sqrt{2}-1\right)i$

u is in the first quadrant

$\operatorname{Arg}(u)=\tan^{-1}\left(\sqrt{2}-1\right)$

$=\frac{\pi}{8}$

iii $u=i+\left(\sqrt{2}-1\right)i^2$

$=1-\sqrt{2}+i$

iu is a rotation of $\frac{\pi}{2}$ anticlockwise from u

so $\operatorname{Arg}(iu)=\operatorname{Arg}\left(1-\sqrt{2}\right)+i$

$=\frac{\pi}{2}+\frac{\pi}{8}$

$=\frac{5\pi}{8}$

b i $\tan\left(\frac{7\pi}{12}\right)=\tan\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)$

$=\frac{\tan\left(\frac{3\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$

$=\frac{-1-\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}}$

$$=\frac{\frac{-3-\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}$$
$$=\frac{-3-\sqrt{3}}{3-\sqrt{3}}\times\frac{-3+\sqrt{3}}{3+\sqrt{3}}$$
$$=\frac{-9-6\sqrt{3}+3}{9-3}$$
$$=-\left(\sqrt{3}+2\right)$$

ii Let $u=-1+\left(\sqrt{3}+2\right)i$

u is in the second quadrant

$$\text{Arg}(u)=\text{Arg}\left(-1+\left(\sqrt{3}+2\right)i\right)=\frac{7\pi}{12}$$

iii $iu=-i+\left(\sqrt{3}+2\right)i^2$
$=-\left(\sqrt{3}+2\right)-i$

$i^2u=1-\left(\sqrt{3}+2\right)i$ is a rotation of 180° anticlockwise

$$\text{so Arg}\left(1-\left(\sqrt{3}+2\right)i\right)=\frac{7\pi}{12}+\pi-2\pi=-\frac{5\pi}{12}$$

iv $i^3u=-iu$
$=\sqrt{3}+2+i$

Is a rotation of 270° anticlockwise

$$\text{so Arg}\left(\sqrt{3}+2+i\right)=\frac{7\pi}{12}+\frac{3\pi}{2}-2\pi=\frac{\pi}{12}$$

24 a $0=(1+i)^n+(1-i)^n$

$$0=\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^n+\left(\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)\right)^n$$
$$0=\left(\sqrt{2}\right)^n\text{cis}\left(\frac{n\pi}{4}\right)+\left(\sqrt{2}\right)^4\text{cis}\left(\frac{-n\pi}{4}\right)$$
$$0=\left(\sqrt{2}\right)^n\left(\cos\left(\frac{n\pi}{4}\right)+i\sin\left(\frac{n\pi}{4}\right)+\cos\left(\frac{n\pi}{4}\right)+i\sin\left(\frac{-n\pi}{4}\right)\right)$$
$$0=2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)$$
$$\Rightarrow\cos\left(\frac{n\pi}{4}\right)=0$$
$$\frac{n\pi}{4}=(2k\pi)\frac{\pi}{2}$$
$$n=2(2k+1)\qquad k\in Z$$

b $(1+i)^n-(1-i)^n=0$

$$2\left(\sqrt{2}\right)^n i\sin\left(\frac{n\pi}{4}\right)=0$$
$$\sin\left(\frac{n\pi}{4}\right)=0$$
$$\frac{n\pi}{4}=k\pi$$
$$n=4k\qquad k\in Z$$

c $0=\left(\sqrt{3}+i\right)^n-\left(\sqrt{3}-i\right)^n$

$$0=\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)\right)^n+\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^n$$

$$0=2^n\ \text{cis}\left(\frac{n\pi}{6}\right)-2^n\ \text{cis}\left(\frac{-n\pi}{6}\right)$$

$$0=\left(\sqrt{2}\right)^n\left(\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)-\left(\cos\left(-\frac{n\pi}{6}\right)+i\sin\left(-\frac{n\pi}{6}\right)\right)\right)$$

$$0=2\times 2^n i\sin\left(\frac{n\pi}{6}\right)$$

$$\Rightarrow \sin\left(\frac{n\pi}{6}\right)=0$$

$$\frac{n\pi}{6}=k\pi$$

$$n=6k \qquad k\in Z$$

d $\left(\sqrt{3}+i\right)^n-\left(\sqrt{3}-i\right)^n=0$

$$2\times 2^n\cos\left(\frac{n\pi}{6}\right)=0$$

$$\cos\left(\frac{n\pi}{6}\right)=0$$

$$\frac{n\pi}{6}=(2k+1)\frac{\pi}{2}$$

$$n=3(2k+1) \qquad k\in Z$$

25 $z=\text{cis}(\theta)$

$=\cos(\theta)+i\sin(\theta)$

a $z+1=\cos(\theta)+1+i\sin(\theta)$

$$|z+1|=\sqrt{(1+\cos(\theta))^2+(\sin(\theta))^2}$$

$$=\sqrt{1+\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)}$$

$$=\sqrt{2+2\cos(\theta)}$$

$$=\sqrt{2(1+\cos(\theta))}$$

$$=\sqrt{2\times 2\cos^2\left(\frac{\theta}{2}\right)}$$

$$=2\cos\left(\frac{\theta}{2}\right) \qquad \text{shown}$$

b $\text{Arg}(1+z)=\tan^{-1}\left(\frac{\sin(\theta)}{1+\cos(\theta)}\right)$

$$=\tan^{-1}\left(\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)$$

$$=\left(\frac{\theta}{2}\right) \qquad \text{shown}$$

c $\frac{1+z}{1-z}=\frac{1+\cos(\theta)+i\sin(\theta)}{1-\cos(\theta)-i\sin(\theta)}\times\frac{1-\cos(\theta)+i\sin(\theta)}{1-\cos(\theta)+i\sin(\theta)}$

$$=\frac{(1+\cos(\theta))(1-\cos(\theta))+i^2\sin^2(\theta)+i\left(\sin(\theta)(1-\cos\theta)+\sin(\theta)(1+\cos(\theta))\right)}{(1-\cos(\theta))^2-i^2\sin^2(\theta)}$$

$$=\frac{1-\cos^2\theta-\sin^2(\theta)+i\sin(\theta)(1-\cos(\theta)+1+\cos(\theta))}{(1-\cos(\theta))^2+\sin^2(\theta)}$$

$$=\frac{i\sin(\theta)\times 2}{1-2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)}$$

$$=\frac{2i\sin(\theta)}{2-2\cos(\theta)}$$

$$=\frac{4i\times\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\left(2\sin^2\left(\frac{\theta}{2}\right)\right)}$$

$$=i\cot\left(\frac{\theta}{2}\right) \quad \text{shown}$$

26 $z=\text{cis}(\theta)=\cos(\theta)+i\sin(\theta)$

a $z^2=\text{cis}(2\theta)=(\cos(\theta)+i\sin(\theta))^2$

$=\cos^2(\theta)+2\cos(\theta)\sin(\theta)i+i^2\sin^2(\theta)$

$=\cos^2(\theta)-\sin^2(\theta)+i2\cos(\theta)\sin(\theta)$

$=\cos(2\theta)+i\sin(2\theta)$

i Re $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$

ii Im $\sin(2\theta)=2\sin(\theta)\cos(\theta)$ shown

b $z^3=\text{cis}(3\theta)=(\cos(\theta)+i\sin(\theta))^3$

$=\cos^3(\theta)+3\cos^2(\theta)i\sin(\theta)+3\cos(\theta)i^2\sin^2(\theta)+i^3\sin^3(\theta)$

$=\cos^3(\theta)+3i\cos^2(\theta)\sin(\theta)-3\cos(\theta)\sin^2(\theta)-i\sin^3(\theta)$

$=\cos^3(\theta)-3\cos(\theta)\sin^2(\theta)+i\left(3\cos^2(\theta)\sin(\theta)-\sin^3(\theta)\right)$

$=\cos(3\theta)+i\sin(3\theta)$

i Re: $\cos(3\theta)=\cos^3(\theta)-3\cos(\theta)\sin^2(\theta)$

$=\cos^3(\theta)-3\cos(\theta)\left(1-\cos^2(\theta)\right)$

$=\cos^3(\theta)-3\cos(\theta)+3\cos^2(\theta)$

$=4\cos^3(\theta)-3\cos(\theta)$

ii Im: $\sin(3\theta)=3\cos^2(\theta)\sin(\theta)-\sin^3(\theta)$

$=3\sin(\theta)\left(1-\sin^2(\theta)-\sin^3(\theta)\right)$

$=3\sin(\theta)-3\sin^3(\theta)-\sin^3(\theta)$

$=3\sin(\theta)-4\sin^3(\theta)$ shown

Exercise 3.4 — Solving polynomial equations

1 $\alpha=-3-4i$

$\beta=-3+4i$

$\alpha+\beta=-6$

$\alpha\beta=9-16i^2$

$=25$

$P(z)=z^2+6z+25$

2 $\alpha=-2i$

$\beta=2i$

$\alpha+\beta=0$

$\alpha\beta=-4i^2$

$=4$

$P(z)=z^2+4$

3 $P(z)=z^3+6z^2+9z-50$

$P(1)=1+6+9-50\neq 0$

$P(2)=8+24+18-50=0$

$z-2$ is a factor

$z^3+6z^2+9z-50=0$

$(z-2)\left(z^2+8z+25\right)=0$

$(z-2)\left(z^2+4z+16+9\right)=0$

$(z-2)\left(z^2+4\right)-9i^2=0$

$(z-2)(z+4-3i)(z+4+3i)=0$

$z=2,-4\pm 3i$

4 $P(z)=z^3-3z^2+4z-12$

$P(1)=1-3+4-12\neq 0$

$P(2)=8-12+18-12\neq 0$

$P(3)=27-27+12-12=0$

$z-3$ is a factor